MATH 303 - MEASURES AND INTEGRATION
FINAL EXAM STUDY GUIDE
SOLUTIONS TO PRACTICE PROBLEMS

Understanding main definitions and theorems. 2 required problems of this type will appear on the exam. Possible additional problems of this kind are to prove one of the results marked with asterisks.

Problem 1.

(a) State the monotone convergence theorem.
(b) State the dominated convergence theroem.
(c) Use the dominated convergence theorem to give a proof of the monotone convergence theorem. [**Hint:** If $\sup_{n\in\mathbb{N}} \int_X |f_n| \, d\mu < \infty$, then the set $\{x \in X : f_n(x) \neq 0 \text{ for some } n\}$ is a σ-finite set. Use this to reduce to the case that the measure space is finite.]

Solution: **(a)** See Theorem 3.10 in the lecture notes.

(b) See Theorem 3.28.

(c) Let $0 \le f_1 \le f_2 \le \dots$ be measurable functions and $f = \lim_{n\to\infty} f_n$. By monotonicity of the integral, $\lim_{n\in\mathbb{N}} \int_X f_n \, d\mu$ exists and is bounded above by $\int_X f \, d\mu$. Hence, if $\sup_{n\in\mathbb{N}} \int_X f_n \, d\mu = \infty$, then we also have $\int_X f \, d\mu = \infty$, and there is nothing to prove.

Assume $\sup_{n\in\mathbb{N}} \int_X f_n \, d\mu = M < \infty$. We want to show $\int_X f \, d\mu = M$. Let $X_0 = \{x \in X : f_n(x) \neq 0 \text{ for some } n \in \mathbb{N}\}$. Note that

$$X_0 = \bigcup_{n,m\in\mathbb{N}} \left\{ f_n > \frac{1}{m} \right\}$$

and

$$\mu\left(\left\{ f_n > \frac{1}{m} \right\} \right) \le m \int_X f_n \, d\mu \le mM < \infty$$

by Markov's inequality. Thus, X_0 is a σ-finite set. Moreover, $f, f_n = 0$ on $X \setminus X_0$, so we may focus our attention on the set X_0.

Write $X_0 = \bigcup_{k\in\mathbb{N}} X_k$ as an increasing union of measurable sets $X_1 \subseteq X_2 \subseteq \dots$ with $\mu(X_k) < \infty$. Let $M_k = \sup_{n\in\mathbb{N}} \int_{X_k} f_n \, d\mu$. We claim that $M_k = \int_{X_k} f \, d\mu$. By monotonicity, $M_k \le \int_{X_k} f \, d\mu$, so it suffices to show $\int_{X_k} f \, d\mu \le M_k$. Let $s : X_k \to [0,\infty)$ be a simple function, $0 \le s \le f$. Define $g_n = \min\{f_n, s\}$ so that $0 \le g_1 \le g_2 \le \dots$ and $\lim_{n\to\infty} g_n = s$. The simple function s is integrable since $\mu(X_k) < \infty$, so $\int_{X_k} g_n \, d\mu \to \int_{X_k} s \, d\mu$ by the dominated convergence theorem. On the other hand, by monotonicity of the integral, $\int_{X_k} g_n \, d\mu \le \int_{X_k} f_n \, d\mu \le M_k$. Hence, $\int_{X_k} s \, d\mu \le M_k$. Taking a supremum over simple functions $0 \le s \le f$, we have $\int_{X_k} f \, d\mu \le M_k$ by the definition of the integral.

Now we turn to the integral over X_0. Let $s : X \to [0,\infty)$ be a simple function, $0 \le s \le f$. We may write $s = \sum_{j=1}^m c_j \mathbb{1}_{E_j}$ for some sets $E_1, \dots, E_m \in \mathcal{B}$ with $E_j \subseteq X_0$ and numbers

$c_1, \ldots, c_m \geq 0$. Then

$$\int_{X_k} s \; d\mu = \sum_{j=1}^{m} c_j \mu(E_j \cap X_k),$$

so

$$\int_X s \; d\mu = \sum_{j=1}^{m} c_j \mu(E_j) = \lim_{k\to\infty} \int_{X_k} s \; d\mu$$

by continuity of μ from below. Therefore, by monotonicity of the integral,

$$\int_X s \; d\mu \leq \lim_{k\to\infty} M_k.$$

Taking a supremum over simple functions $0 \leq s \leq f$,

$$\int_X f \; d\mu \leq \lim_{k\to\infty} M_k.$$

It remains to check that $M = \lim_{k\to\infty} M_k$. Note that for each $n \in \mathbb{N}$, since $\int_X f_n \; d\mu \leq M < \infty$, the dominated convergence theorem implies

$$\int_X f_n \; d\mu = \lim_{k\to\infty} \int_{X_k} f_n \; d\mu.$$

Hence,

$$M = \sup_{n\in\mathbb{N}} \int_X f_n \; d\mu = \sup_{n\in\mathbb{N}} \sup_{k\in\mathbb{N}} \int_{X_k} f_n \; d\mu = \sup_{k\in\mathbb{N}} \sup_{n\in\mathbb{N}} \int_{X_k} f_n \; d\mu = \lim_{k\to\infty} M_k$$

by the principle of iterated suprema. Thus, $\int_X f \; d\mu \leq M$ as desired.

Problem 2. Let $(X, \mathcal{B})$, $(Y, \mathcal{C})$, and $(Z, \mathcal{D})$ be measurable spaces.

(a) Show that $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} = \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$ and that this σ-algebra is equal to the σ-algebra on $X \times Y \times Z$ generated by the family of "measurable boxes" $\{B \times C \times D : B \in \mathcal{B}, C \in \mathcal{C}, D \in \mathcal{D}\}$.

(b) Suppose $\mu : \mathcal{B} \to [0, \infty]$, $\nu : \mathcal{C} \to [0, \infty]$, and $\rho : \mathcal{D} \to [0, \infty]$ are σ-finite measures. Show that $(\mu \times \nu) \times \rho = \mu \times (\nu \times \rho)$ and that this measure is the unique measure on $\mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D}$ assigning a measure of $\mu(B)\nu(C)\rho(D)$ to each measurable box $B \times C \times D$.

Solution: **(a)** Let $\mathcal{F} = \{B \times C \times D : B \in \mathcal{B}, C \in \mathcal{C}, D \in \mathcal{D}\}$ be the family of measurable boxes. Writing $B \times C \times D = (B \times C) \times D = B \times (C \times D)$, we see that $\mathcal{F} \subseteq (\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}$ and $\mathcal{F} \subseteq \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$. It therefore suffices to show $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}, \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}) \subseteq \sigma(\mathcal{F})$. We will show $(B \otimes \mathcal{C}) \otimes \mathcal{D} \subseteq \sigma(\mathcal{F})$. The other inclusion is proved in exactly the same way.

Recall that $(B \otimes \mathcal{C}) \otimes \mathcal{D}$ is defined to be the σ-algebra generated by the family of measurable rectangles $\mathcal{R} = \{E \times D : E \in B \otimes \mathcal{C}, D \in \mathcal{D}\}$, so it suffices to prove $\mathcal{R} \subseteq \sigma(\mathcal{F})$. Let $\mathcal{F}' = \{E \in B \otimes \mathcal{C} : E \times D \in \sigma(\mathcal{F}) \text{ for every } D \in \mathcal{D}\}$. Then $\mathcal{F}'$ contains all measurable rectangles $B \times C$ with $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Moreover, since $\sigma(\mathcal{F})$ is a σ-algebra, the family $\mathcal{F}'$ is also a σ-algebra (the set operations are preserved under product with $D \in \mathcal{D}$). Thus, $\mathcal{F}' = \mathcal{B} \otimes \mathcal{C}$. This proves $\sigma(\mathcal{F}) \supseteq \mathcal{R}$ as desired.

(b) We will apply the π-λ theorem. Note that the family of measurable boxes is a π-system on $X \times Y \times Z$:

$$(B_1 \times C_1 \times D_1) \cap (B_2 \times C_2 \times D_2) = (B_1 \cap B_2) \times (C_1 \cap C_2) \times (D_1 \cap D_2).$$

Moreover, given a measurable box $B \times C \times D$, the definition of a product measure provides an equality

$$\begin{aligned}((\mu \times \nu) \times \rho)(B \times C \times D) &= (\mu \times \nu)(B \times C)\rho(D) = \mu(B)\nu(C)\rho(D) \\ &= \mu(B)(\nu \times \rho)(C \times D) = (\mu \times (\nu \times \rho))(B \times C \times D),\end{aligned}$$

so $(\mu \times \nu) \times \rho$ and $\mu \times (\nu \times \rho)$ are both examples of measures assigning a value of $\mu(B)\nu(C)\rho(D)$ to every measurable box $B \times C \times D$.

Let $\omega_1, \omega_2 : \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D} \to [0, \infty]$ be any two measures assigning a measure of $\mu(B)\nu(C)\rho(D)$ to each measurable box $B \times C \times D$. We want to show $\omega_1 = \omega_2$.

Since μ, ν, and ρ are σ-finite, we may write $X = \bigcup_{n \in \mathbb{N}} X_n$, $Y = \bigcup_{n \in \mathbb{N}} Y_n$, and $Z = \bigcup_{n \in \mathbb{N}} Z_n$ with $X_1 \subseteq X_2 \subseteq \dots$, $Y_1 \subseteq Y_2 \subseteq \dots$, $Z_1 \subseteq Z_2 \subseteq \dots$, and $\mu(X_n), \nu(Y_n), \rho(Z_n) < \infty$ for every $n \in \mathbb{N}$. Let $A_n = X_n \times Y_n \times Z_n$ for each $n \in \mathbb{N}$. Then A_n is a measurable box, so

$$\omega_1(A_n) = \mu(X_n)\nu(Y_n)\rho(Z_n) = \omega_2(A_n). \tag{1}$$

For each $n \in \mathbb{N}$, consider the family

$$\mathcal{L}_n = \{E \in \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D} : \omega_1(E \cap A_n) = \omega_2(E \cap A_n)\},$$

and let $\mathcal{L} = \bigcap_{n \in \mathbb{N}} \mathcal{L}_n$. For each $n \in \mathbb{N}$, the family $\mathcal{L}_n$ is a λ-system, since the measures $\omega_{1,n}(E) = \omega_1(E \cap A_n)$ and $\omega_{2,n}(E) = \omega_2(E \cap A_n)$ are finite measures on $(X \times Y \times Z, \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D})$ with $\omega_{1,n}(X \times Y \times Z) = \omega_{2,n}(X \times Y \times Z)$ by (1) and $\mathcal{L}_n = \{E \in \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D} : \omega_{1,n}(E) = \omega_{2,n}(E)\}$. (We showed that such families of sets are always λ-systems in the exercises; see P5 on Exercise Sheet 5.) The intersection of λ-systems is again a λ-system, so $\mathcal{L}$ is a λ-system on $X \times Y \times Z$ containing the measurable boxes. By the π-λ theorem, it follows that $\mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D} \subseteq \mathcal{L}$. Then by continuity from below of ω_1 and ω_2, we have

$$\omega_1(E) = \lim_{n \to \infty} \omega_1(E \cap A_n) = \lim_{n \to \infty} \omega_2(E \cap A_n) = \omega_2(E)$$

for every $E \in \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D}$. That is, $\omega_1 = \omega_2$.

Problem 3. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f \in L^1(\mu)$. Show that the following are equivalent:

(i) $f = 0$ a.e.;
(ii) $\int_X |f| \, d\mu = 0$;
(iii) $\int_E f \, d\mu = 0$ for every $E \in \mathcal{B}$.

Solution: See P1 on Exercise Sheet 4 on Moodle.

Applying the main theorems. 6 problems of this type will appear on the exam, out of which you may choose which 4 to solve.

Problem 4. Let λ be the Lebesgue measure on $\mathbb{R}$. For $\varepsilon > 0$, let

$$A_\varepsilon = \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}} \text{ for infinitely many } p, q \in \mathbb{Z} \text{ with } q \geq 1 \right\}.$$

Show that $\lambda(A_\varepsilon) = 0$ for every $\varepsilon > 0$.

[Note: A theorem of Dirichlet, which can be proved using the pigeonhole principle, says that every irrational number x can be approximated by rationals in such a way that $\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$ for infinitely many $p, q \in \mathbb{Z}$ with $q \geq 1$. This problem shows that the exponent 2 is best possible for almost all numbers.]

Solution: For $q \in \mathbb{N}$, let

$$A_{\varepsilon,q} = \bigcup_{p=0}^{q} \left(\frac{p}{q} - \frac{1}{q^{2+\varepsilon}}, \frac{p}{q} + \frac{1}{q^{2+\varepsilon}} \right).$$

Then $A_\varepsilon = \{x \in [0,1] : x \in A_{\varepsilon,q} \text{ for infinitely many } q \in \mathbb{N}\}$. By direct computation, $\lambda(A_{\varepsilon,q}) \leq (q+1)\frac{2}{q^{2+\varepsilon}} = \frac{2}{q^{1+\varepsilon}} + \frac{1}{q^{2+\varepsilon}}$, so $\sum_{q=1}^{\infty} \lambda(A_{\varepsilon,q}) < \infty$. Hence, by the Borel–Cantelli lemma, $\lambda(A_\varepsilon) = 0$.

Problem 5. Let $a \in (0,1)$ and $\varepsilon > 0$.

(a) Show that there exists $M \in \mathbb{N}$ with the following property: if $(X, \mathcal{B}, \mu)$ is a probability space and $(A_n)_{n\in\mathbb{N}}$ is a sequence of measurable sets such that $\inf_{n\in\mathbb{N}} \mu(A_n) = a$, then there exist $1 \leq n < m \leq M$ such that $\mu(A_n \cap A_m) > a^2 - \varepsilon$.

(b) Prove the following generalization for intersections of more sets. Let $k \in \mathbb{N}$. Show that there exists $M_k \in \mathbb{N}$ (depending also on a and ε) with the property: if $(X, \mathcal{B}, \mu)$ is a probability space and $(A_n)_{n\in\mathbb{N}}$ is a sequence of measurable sets such that $\inf_{n\in\mathbb{N}} \mu(A_n) = a$, then there exist $1 \leq n_1 < n_2 < \cdots < n_k \leq M_k$ such that

$$\mu\left(\bigcap_{j=1}^{k} A_{n_j} \right) > a^k - \varepsilon.$$

Solution: Part (a) is a special case of (b), so we will go directly to proving (b). Fix a large value of M to be determined later. Given a probability space $(X, \mathcal{B}, \mu)$ and measurable sets $(A_n)_{n\in\mathbb{N}}$ with $\inf_{n\in\mathbb{N}} \mu(A_n) = a$, define new sets $B_n = \left\{ \sum_{m=1}^{M} \mathbb{1}_{A_m} = n \right\}$ for $0 \leq n \leq M$. Then for any k, we have

$$\sum_{m=0}^{M} \binom{m}{k} \mu(B_m) = \sum_{1 \leq m_1 < \cdots < m_k \leq M} \mu(A_{m_1} \cap \cdots \cap A_{m_k}). \tag{2}$$

Since μ is a probability space and $B_0 \sqcup B_1 \sqcup \cdots \sqcup B_m = X$, the numbers $t_m = \mu(B_m)$ are nonnegative and sum to 1. Moreover, $x \mapsto \binom{x}{k}$ is a convex function, so by Jensen's inequality,

$$\sum_{m=0}^{M} \binom{m}{k} \mu(B_m) \geq \binom{\sum_{m=0}^{M} m\mu(B_m)}{k} \geq \binom{Ma}{k}.$$

There are $\binom{M}{k}$ terms on the right hand size of (2), so the average size of a k-fold intersection is at least

$$\frac{\binom{Ma}{k}}{\binom{M}{k}} = \frac{Ma(Ma-1)\dots(Ma-k+1)}{M(M-1)\dots(M-k+1)}.$$

For fixed a and k, this ratio converges to a^k as $M \to \infty$. Thus, choosing M large enough (depending only on a, k, and ε), we can ensure

$$\frac{\binom{Ma}{k}}{\binom{M}{k}} > a^k - \varepsilon.$$

This controls the average size of k-fold intersections, so there must be at least one k-fold intersection for which the desired inequality holds.

Problem 6. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ a measurable function. Prove Chebyshev's inequality: for every $c \in (0, \infty)$ and $p \in [1, \infty)$,

$$\mu(\{|f| > c\}) \leq \left(\frac{\|f\|_p}{c} \right)^p$$

Solution: Let $g = c\mathbb{1}_{\{|f|>c\}}$. Then $|f| \geq g$, so by monotonicity of the integral,

$$\|f\|_p^p = \int_X |f|^p \, d\mu \geq \int_X g^p \, d\mu = c^p \mu(\{|f| > c\}).$$

Dividing both sides by c^p gives Chebyshev's inequality.

Problem 7. Let $E \subseteq \mathbb{R}$ be a Lebesgue-measurable set with $\lambda(E) > 0$. Fix a finite set $F \subseteq \mathbb{R}$. Show that E contains a homothetic copy of F, i.e. a set of the form $aF + b = \{af + b : f \in F\}$ with $a \neq 0$ and $b \in \mathbb{R}$.

[Note: There is a deep theorem in additive combinatorics, known as Szemerédi's theorem, that provides a strengthening to the conclusion; namely, one can bound the scaling factor $a > \delta$ for some δ depending on the set F and the size of the set E.]

Solution: We first prove a lemma using Littlewood's first principle.

Lemma. For any $c < 1$, there exists a bounded open interval $I \subseteq \mathbb{R}$ such that $\lambda(E \cap I) > c\lambda(I)$.

Proof of Lemma. Let $c' = \frac{2c}{1+c}$ so that $\frac{c'}{2-c'} = c$. By continuity of λ from below, the set $E_N = E \cap [-N, N]$ has $\lambda(E_N) > 0$ for some $N \in \mathbb{N}$. We may therefore assume $0 < \lambda(E) < \infty$. By Littlewood's first principle, there is a finite collection of disjoint open intervals $I_i = (a_i, b_i)$, $i = 1, \ldots, n$, such that

$$\lambda\left(E \triangle \bigsqcup_{i=1}^n I_i\right) < (1 - c')\lambda(E).$$

Therefore, writing $E = (E \cap \bigsqcup_{i=1}^n I_i) \sqcup (E \setminus \bigsqcup_{i=1}^n I_i)$, we have

$$\lambda\left(E \cap \bigsqcup_{i=1}^n I_i\right) > c'\lambda(E). \tag{3}$$

Also, writing $\bigsqcup_{i=1}^n I_i = (E \cap \bigsqcup_{i=1}^n I_i) \sqcup (\bigsqcup_{i=1}^n I_i \setminus E)$, we have

$$\lambda\left(\bigsqcup_{i=1}^n I_i\right) < (2 - c')\lambda(E). \tag{4}$$

In particular, combining (3) and (4),

$$\sum_{i=1}^n \lambda(E \cap I_i) = \lambda\left(E \cap \bigsqcup_{i=1}^n I_i\right) > \frac{c'}{2-c'}\lambda\left(\bigsqcup_{i=1}^n I_i\right) = c\sum_{i=1}^n \lambda(I_i).$$

Hence, for some $i \in \{1, \ldots, n\}$, we must have $\lambda(E \cap I_i) > c\lambda(I_i)$, proving the lemma. □

Let $k = |F|$. Applying the lemma with $c = 1 - \frac{1}{2k}$, let I be an open interval such that $\lambda(E \cap I) > \left(1 - \frac{1}{2k}\right) \lambda(I)$. Suppose $t_1, \dots, t_k \in (-\delta, \delta)$ with $\delta = \frac{\lambda(I)}{2k}$. Then

$$\begin{aligned}\lambda\left(\bigcap_{j=1}^{k}(E - t_j)\right) &\geq \lambda\left(\bigcap_{j=1}^{k}((E \cap I) - t_j)\right) \\ &\geq \lambda\left(\bigcap_{j=1}^{k}(I - t_j)\right) - k\lambda(I \setminus E) \\ &> \lambda\left(\bigcap_{j=1}^{k}\underbrace{(I \cap (I - t_j))}_{\text{measure} > \lambda(I) - \delta}\right) - \frac{\lambda(I)}{2} \\ &> \lambda(I) - k\delta - \frac{\lambda(I)}{2} = 0.\end{aligned}$$

Take $a = \frac{\delta}{\max_{f \in F} |f| + 1}$ so that $|af| < \delta$ for every $f \in F$. Therefore,

$$\lambda\left(\bigcap_{f \in F}(E - af)\right) > 0.$$

In particular, $\bigcap_{f \in F}(E - af) \neq \emptyset$, so let $b \in \bigcap_{f \in F}(E - af)$. Then $af + b \in E$ for every $f \in F$, so we have found a homothetic copy $aF + b$ in E.

Problem 8. For fixed $x \in \mathbb{R}$, let $L_x = \{(x, y) : y \in \mathbb{R}\} \subseteq \mathbb{R}^2$ be the vertical line over x. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the second coordinate $\pi(x, y) = y$. Define

$$\tau = \{G \subseteq \mathbb{R}^2 : \pi(G \cap L_x) \text{ is open for every } x \in \mathbb{R}\}.$$

(a) Show that τ is a topology on $\mathbb{R}^2$ and $(\mathbb{R}^2, \tau)$ is a locally compact Hausdorff space.
(b) Prove that $K \subseteq \mathbb{R}^2$ is compact (with respect to τ) if and only if $\pi(K \cap L_x)$ is compact for every $x \in \mathbb{R}$ and $K \cap L_x = \emptyset$ for all but finitely many x.
(c) Define $\varphi : C_c(\mathbb{R}^2, \tau) \to \mathbb{C}$ by

$$\varphi(f) = \sum_{x \in \mathbb{R}} \int_{\mathbb{R}} f(x, y)\, dy,$$

where the integral with respect to y is the Riemann integral. Show that φ is a positive linear functional.
(d) Determine the measure μ representing φ.

Solution: **(a)** We check that τ satisfies the axioms of a topology:

- $\emptyset \in \tau$: $\pi(\emptyset \cap L_x) = \emptyset$ for every $x \in \mathbb{R}$, and $\emptyset$ is open in $\mathbb{R}$.
- $\mathbb{R}^2 \in \tau$: $\pi(\mathbb{R}^2 \cap L_x) = \mathbb{R}$ for every $x \in \mathbb{R}$ and $\mathbb{R}$ is open in $\mathbb{R}$.
- τ is closed under arbitrary unions: if $(G_i)_{i \in I}$ is a family of elements of τ and $G = \bigcup_{i \in I} G_i$, then $\pi(G \cap L_x) = \bigcup_{i \in I} \pi(G_i \cap L_x)$ is open in $\mathbb{R}$ for every $x \in \mathbb{R}$, so $G \in \tau$.
- τ is closed under finite intersections: if $G_1, G_2 \in \tau$, then $\pi(G_1 \cap G_2 \cap L_x) = \pi(G_1 \cap L_x) \cap \pi(G_2 \cap L_x)$ is open in $\mathbb{R}$ for every $x \in \mathbb{R}$, so $G_1 \cap G_2 \in \tau$.

It remains to check that $(\mathbb{R}^2, \tau)$ is locally compact and Hausdorff. First we show that τ defines a Hausdorff topology on $\mathbb{R}^2$. Let $x, y \in \mathbb{R}^2$, $x \neq y$. Write $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If $x_1 \neq y_1$, then the sets $U = \{x_1\} \times \mathbb{R}$ and $V = \{y_1\} \times \mathbb{R}$ are open neighborhoods of x and y

respectively, and $U \cap V = \emptyset$. If $x_2 \neq y_2$, we take $U = \mathbb{R} \times (x_2 - \delta, x_2 + \delta)$ and $V = \mathbb{R} \times (y_2 - \delta, y_2 + \delta)$ with $\delta < \frac{1}{2}|x_2 - y_2|$.

Finally, τ defines a locally compact topology: given $x = (x_1, x_2) \in \mathbb{R}^2$, the set $U = \{x_1\} \times (x_2 - 1, x_2 + 1)$ is open its closure $K = \{x_1\} \times [x_2 - 1, x_2 + 1]$ is compact. (Compactness of K follows from the argument we present below for part (b)).

(b) Suppose K is compact. We want to show $\pi(K \cap L_x)$ is compact for every $x \in \mathbb{R}$ and $K \cap L_x = \emptyset$ for all but finitely many x.

Fix $x \in \mathbb{R}$, and let $K_x = \pi(K \cap L_x)$. Let $(U_i)_{i \in I}$ be an open cover of K_x in $\mathbb{R}$. Then $K \subseteq \bigcup_{i \in I} (\{x\} \times U_i) \cup (\mathbb{R} \setminus \{x\}) \times \mathbb{R})$. Since K is compact, there is a finite subcover $K \subseteq \bigcup_{j=1}^{n} (\{x\} \times U_{i_j}) \cup (\mathbb{R} \setminus \{x\}) \times \mathbb{R})$. But then $K_x \subseteq \bigcup_{j=1}^{n} U_{i_j}$. Thus, we have extracted a finite subcover of K_x, so K_x is compact.

Now consider the family of sets $U_x = \{x\} \times \mathbb{R}$. Then $(U_x)_{x \in \mathbb{R}}$ is an open cover of K, so there exists a finite set $F \subseteq \mathbb{R}$ such that $K \subseteq \bigcup_{x \in F} U_x$. In other words, $\pi(K \cap L_x) = \emptyset$ for $x \notin F$.

Conversely, suppose K satisfies the two properties: $\pi(K \cap L_x)$ is compact for every $x \in \mathbb{R}$ and $K \cap L_x = \emptyset$ for all but finitely many x. Let $(G_i)_{i \in I}$ be a family of sets in τ such that $K \subseteq \bigcup_{i \in I} G_i$. By assumption, the set $F = \{x \in \mathbb{R} : \pi(K \cap L_x) \neq \emptyset\}$ is finite. Moreover, for each $x \in F$, the set $K_x = \pi(K \cap L_x)$ is compact, and we have an open cover $K_x \subseteq \bigcup_{i \in I} \pi(G_i \cap L_x)$. Hence, there is a finite subcover $K_x \subseteq \bigcup_{i \in J_x} \pi(G_i \cap L_x)$ for some finite set $J_x \subseteq I$. Letting $J = \bigcup_{x \in F} J_x$, we have that J is finite and $K \subseteq \bigcup_{j \in J} G_j$. Thus, K is a compact subset of $(\mathbb{R}^2, \tau)$.

(c) Let $f \in C_c(\mathbb{R}^2, \tau)$, and let $K = \operatorname{supp}(f)$. By (b), the set $F = \{x \in \mathbb{R} : \pi(K \cap L_x) \neq \emptyset\}$ is finite, and $K_x = \pi(K \cap L_x)$ is compact for each $x \in F$. We can therefore evaluate

$$\varphi(f) = \sum_{x \in F} \int_{K_x} f(x, y) \, dy$$

as a finite sum of Riemann integrals. Thus, the *a priori* infinite sum in the definition of φ is in fact finite, so φ is well-defined. Linearity of φ follows from linearity of addition and the Riemann integral, and positivity of φ follows from positivity of the Riemann integral. Therefore φ is a positive linear functional on $C_c(\mathbb{R}^2, \tau)$.

(d) We claim that the measure μ can be evaluated as follows: if $E \subseteq \mathbb{R}^2$ is a Borel set (with respect to τ), then

$$\mu(E) = \begin{cases} \sum_{x \in I} \lambda(E_x), & \text{if } I = \{x \in \mathbb{R} : E_x \neq \emptyset\} \text{ is countable,} \\ \infty, & \text{otherwise.} \end{cases} \tag{5}$$

Here, E_x is the cross-section $E_x = \{y \in \mathbb{R} : (x, y) \in E\} = \pi(E \cap L_x)$, and λ is the Lebesgue measure on $\mathbb{R}$.

Let $\nu(E)$ be the quantity defined by the right hand side of (5). By the uniqueness part of the Riesz representation theorem, it suffices to check that ν is a Radon measure on $(\mathbb{R}^2, \tau)$ and $\int_{\mathbb{R}^2} f \, d\nu = \varphi(f)$ for $f \in C_c(\mathbb{R}^2, \tau)$.

Claim 1: ν is a measure.

If $E = \emptyset$, then $E_x = \emptyset$ for every $x \in \mathbb{R}$, so $\nu(\emptyset) = 0$. Suppose $E_1, E_2, \dots \in \operatorname{Borel}(\mathbb{R}^2, \tau)$ are pairwise disjoint, and let $E = \bigsqcup_{n \in \mathbb{N}} E_n$. Then $E_x = \bigsqcup_{n \in \mathbb{N}} (E_n)_x$ for every $x \in \mathbb{R}$. Let $I_n = \{x \in \mathbb{R} : (E_n)_x \neq \emptyset\}$ for each $n \in \mathbb{N}$, and note that $I = \bigcup_{n \in \mathbb{N}} I_n$. If I_n is uncountable for some $n \in \mathbb{N}$ (in which case $\nu(E_n) = \infty$), then I is uncountable, so $\nu(E) = \infty$. Therefore, $\nu(E) = \sum_{k=1}^{\infty} \nu(E_k)$. On the other hand, if I_n is countable for every $n \in \mathbb{N}$, then I is also countable, and we have

$$\nu(E) = \sum_{x \in I} \lambda(E_x) = \sum_{x \in I} \lambda\left(\bigsqcup_{n \in \mathbb{N}} (E_n)_x \right) = \sum_{x \in I} \sum_{n=1}^{\infty} \lambda\left((E_n)_x\right) = \sum_{n=1}^{\infty} \sum_{x \in I} \lambda\left((E_n)_x\right) = \sum_{n=1}^{\infty} \nu(E_n).$$

The interchange in the order of summation is justified by the fact that all of the quantities $\lambda\left((E_n)_x\right)$ are nonnegative. This proves that ν is a measure.

<u>Claim 2</u>: ν is locally finite.

Let $K \subseteq \mathbb{R}^2$ be compact (with respect to τ). Then by part (b), the set $I = \{x \in \mathbb{R} : K_x \neq \emptyset\}$ is finite, and K_x is compact for every $x \in I$. Hence, by local finiteness of the Lebesgue measure λ, we have that

$$\lambda(K) = \sum_{x \in I} \lambda(K_x)$$

is a finite sum of finite numbers, so $\lambda(K) < \infty$.

<u>Claim 3</u>: ν is outer regular.

Let $E \in \text{Borel}(\mathbb{R}^2, \tau)$ be an arbitrary Borel set. We want to show $\nu(E) = \inf_{U \supseteq E \text{ open}} \nu(U)$. Monotonicity of ν produces the inequality $\nu(E) \leq \inf_{U \supseteq E \text{ open}} \nu(U)$, so it suffices to check $\nu(E) \geq \inf_{U \supseteq E \text{ open}} \nu(U)$. If $\nu(E) = \infty$, there is nothing to check, so assume $\nu(E) < \infty$. That is, $I = \{x \in \mathbb{R} : E_x \neq \emptyset\}$ is countable and $\nu(E) = \sum_{x \in I} \lambda(E_x) < \infty$. Let $\varepsilon > 0$. For each $x \in I$, let $\varepsilon_x > 0$ such that $\sum_{x \in I} \varepsilon_x = \varepsilon$. (This is made possible by the fact that I is a countable set.) By outer regularity of the Lebesgue measure, we may choose, for each $x \in I$, an open set $U_x \supseteq E_x$ such that $\lambda(U_x) < \lambda(E_x) + \varepsilon_x$. Let $U = \bigsqcup_{x \in I} (\{x\} \times U_x)$. Then $U \supseteq E$ is open in $(\mathbb{R}^2, \tau)$ and $\nu(U) = \sum_{x \in I} \lambda(U_x) < \nu(E) + \varepsilon$. Thus, $\nu(E) \geq \inf_{U \supseteq E \text{ open}} \nu(U)$ as desired.

<u>Claim 4</u>: ν is inner regular on open sets.

Let $U \subseteq \mathbb{R}^2$ be open (with respect to τ). We want to show $\nu(U) = \sup_{K \subseteq U \text{ compact}} \nu(K)$. By the definition of the topology τ, the set U_x is open for every $x \in \mathbb{R}$.

First suppose $I = \{x \in \mathbb{R} : U_x \neq \emptyset\}$ is countable. Then

$$\nu(U) = \sum_{x \in I} \lambda(E_x) = \sup_{F \subseteq I \text{ finite}} \sum_{x \in F} \lambda(E_x).$$

Let $F \subseteq I$ be an arbitrary finite subset. Let $\varepsilon > 0$ be arbitrary. By inner regularity of the Lebesgue measure, pick a compact set $K_x \subseteq E_x$ with $\lambda(K_x) > \lambda(E_x) - \frac{\varepsilon}{|F|}$ for each $x \in F$. Then $K = \bigsqcup_{x \in F} (\{x\} \times K_x)$ is compact in $(\mathbb{R}^2, \tau)$ by part (b), and

$$\nu(K) = \sum_{x \in F} \lambda(K_x) > \sum_{x \in F} \lambda(E_x) - \varepsilon.$$

Thus, $\sup_{K \subseteq U \text{ compact}} \nu(K) \geq \nu(U)$.

Now suppose $I = \{x \in \mathbb{R} : U_x \neq \emptyset\}$ is uncountable. In this case, we want to show $\sup_{K \subseteq U \text{ compact}} \nu(K) = \infty$. Note that $\lambda(U_x) > 0$ for every $x \in I$, since every open subset of the real line has positive Lebesgue measure. Therefore, $I_n = \{x \in \mathbb{R} : \lambda(U_x) > \frac{1}{n}\}$ is infinite (in fact, uncountable) for some $n \in \mathbb{N}$. By inner regularity of the Lebesgue measure, there exist compact sets $K_x \subseteq U_x$ for each $x \in I_n$ with the property $\lambda(K_x) > \frac{1}{2n}$. Given any finite set $F \subseteq I$, the set $K_F = \bigsqcup_{x \in F} (\{x\} \times K_x)$ is a compact subset of U, and

$$\nu(K_F) = \sum_{x \in F} \lambda(K_x) \geq \frac{1}{2n} |F|.$$

Therefore, $\sup_{K \subseteq U \text{ compact}} \nu(K) = \infty$ as desired.

Claims 1–4 together show that ν is a Radon measure. The next claim therefore completes the proof.

Claim 5: For every $f \in C_c(X)$,

$$\int_{\mathbb{R}^2} f \, d\nu = \varphi(f).$$

Decomposing f into its real and imaginary parts and then each of these into positive and negative parts, we may assume that f is a nonnegative real-valued function. Let $K = \text{supp}(f)$, and let $I = \{x \in \mathbb{R} : K_x \neq \emptyset\}$. By (b), I is a finite set and K_x is compact for every $x \in I$. Let $0 \le s_1 \le s_2 \le \dots$ be a sequence of simple functions increasing to f. We may write $s_n = \sum_{j=1}^{k_n} c_{n,j} \mathbb{1}_{E_{n,j}}$ with $c_{n,j} \ge 0$ and $E_{n,j} \subseteq K$, since f vanishes outside of K and $0 \le s_n \le f$. Then

$$\int_{\mathbb{R}^2} s_n \, d\nu = \sum_{j=1}^{k_n} c_{n,j} \nu(E_{n,j}) = \sum_{j=1}^{k_n} c_{n,j} \sum_{x \in I} \lambda\left((E_{n,j})_x\right) = \sum_{x \in I} \sum_{j=1}^{k_n} c_{n,j} \lambda\left((E_{n,j})_x\right) = \sum_{x \in I} \int_{\mathbb{R}} (s_n)_x \, d\lambda.$$

Applying the monotone convergence theorem twice, we conclude

$$\int_{\mathbb{R}^2} f \, d\nu = \lim_{n\to\infty} \int_{\mathbb{R}^2} s_n \, d\nu = \lim_{n\to\infty} \sum_{x \in I} \int_{\mathbb{R}} s_n(x,y) \, d\lambda(y) = \sum_{x \in I} \int_{K_x} f(x,y) \, dy = \varphi(f).$$

Problem 9. For $x \in [0,1)$, consider the binary expansion $x = \sum_{j=1}^{\infty} \frac{a_j(x)}{2^j}$ with $a_j(x) \in \{0,1\}$. Let $f(x) = \min\{j \in \mathbb{N} : a_j(x) = 1\}$.

(a) Show that f is Borel-measurable.

(b) Compute the integral of f with respect to the Lebesgue measure on $[0,1)$.

[Note: This problem has a probabilistic interpretation. Sampling $x \in [0,1)$ randomly according to the Lebesgue measure, the sequence $a_1(x), a_2(x), a_3(x), \dots$ is a sequence of independent fair coin flips (where we interpret 0 as tails and 1 as heads). With this interpretation, the value of $\int_{[0,1)} f \, d\lambda$ is the expected number of flips required until we see a result of heads.]

Solution: **(a)** The function f may be rewritten as

$$f = \mathbb{1}_{[1/2,1)} + 2\mathbb{1}_{[1/4,1/2)} + \cdots = \sum_{n=1}^{\infty} n \mathbb{1}_{[2^{-n}, 2^{-(n-1)})},$$

which is the supremum of a sequence of measurable simple functions, hence measurable.

(b) By the monotone convergence theorem and the series expression for f,

$$\int_0^1 f(x) \, dx = \sum_{n=0}^{\infty} n \left(2^{-(n-1)} - 2^{-n}\right) = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

To compute the value of this sum, we consider the power series $g(x) = \sum_{n=1}^{\infty} n x^n$. Note that $g(x)$ converges absolutely for $|x| < 1$, so we may differentiate term by term to obtain the expression

$$g(x) = x \sum_{n=1}^{\infty} n x^{n-1} = x \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}.$$

Substituting $x = \frac{1}{2}$, we conclude

$$\int_0^1 f(x)\ dx = 2.$$

Problem 10. Let $(X, \mathcal{B}, \mu)$ be a measure space. Show that the following are equivalent:

(i) μ is s-finite;
(ii) there exists a finite measure ν such that $\mu \approx \nu$;
(iii) there exists a σ-finite measure ν such that $\mu \ll \nu$.

Solution: (i) $\implies$ (ii). This is essentially contained in the proof of the Lebesgue decomposition theorem in the lecture notes. Write $\mu = \sum_{n=1}^{\infty} \mu_n$ with $\mu_n(X) < \infty$, and define $\nu = \sum_{n=1}^{\infty} \frac{\mu_n}{2^n(\mu_n(X)+1)}$. Then $\nu(X) < 1$ and $\mu \ll \nu$.

(ii) $\implies$ (iii). Finite measures are σ-finite, so this implication is immediate.

(iii) $\implies$ (i). Let ν be a σ-finite measure such that $\mu \ll \nu$. Then by the Radon–Nikodym theorem, there exists a Radon–Nikodym derivative $f = \frac{d\mu}{d\nu}$. Moreover, since ν is σ-finite, we may decompose $X = \bigsqcup_{n\in\mathbb{N}} X_n$ with $\nu(X_n) < \infty$. For $m \geq 0$, let

$$f_{m,n} = \left((f-m)\mathbb{1}_{\{m<f\leq m+1\}} + \mathbb{1}_{\{f>m+1\}}\right)\mathbb{1}_{X_n}.$$

Then $0 \leq f_{m,n} \leq 1$, $\int_X f_{m,n}\ d\nu \leq \nu(X_n) < \infty$, and

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} f_{m,n} = \sum_{m=0}^{\infty}\left((f-m)\mathbb{1}_{\{m<f\leq m+1\}} + \mathbb{1}_{\{f>m+1\}}\right) = f.$$

Therefore, if we define measures $\mu_{m,n}$ by $d\mu_{m,n} = f_{m,n}\ d\nu$, then $\mu_{m,n}$ is a finite positive measure for every m, n, and $\mu = \sum_{m,n} \mu_{m,n}$, so μ is s-finite.

Problem 11. Let μ_1, μ_2 be finite positive measures on a measurable space $(X, \mathcal{B})$. Characterize the pairs (μ_1, μ_2) for which $(\mu_1 - \mu_2)^+ = \mu_1$ and $(\mu_1 - \mu_2)^- = \mu_2$.

Solution: Let $\mu = \mu_1 - \mu_2$. We claim that $\mu^+ = \mu_1$ and $\mu^- = \mu_2$ if and only if $\mu_1 \perp \mu_2$.

First if $\mu_1 \perp \mu_2$, then μ_1 and μ_2 are mutually singular positive measures such that $\mu = \mu_1 - \mu_2$, so by uniqueness of the Jordan decomposition, $\mu_1 = \mu^+$ and $\mu_2 = \mu^-$.

Conversely, suppose $\mu^+ = \mu_1$ and $\mu^- = \mu_2$. By the Jordan decomposition theorem, the measures μ^+ and μ^- are mutually singular, so $\mu_1 \perp \mu_2$.

Problem 12. Let $(X, \mathcal{B}, \mu)$ be a finite measure space, and let $A, B \in \mathcal{B}$. Define $\nu(E) = \mu(E \cap A) - \mu(E \cap B)$ for $E \in \mathcal{B}$.

(a) Show that ν is a signed measure.
(b) Determine the Hahn decomposition of ν.
(c) Show that $\nu \ll \mu$.
(d) Compute the Radon–Nikodym derivative $\frac{d\nu}{d\mu}$.

Solution: (a) Since μ is a finite measure, the difference $\mu(E\cap A) - \mu(E\cap B)$ is well-defined. We must check $\nu(\emptyset) = 0$ and ν is countable additive. Both properties follow from the corresponding properties of the measure μ:

$$\nu(\emptyset) = \mu(\emptyset \cap A) - \mu(\emptyset \cap B) = \mu(\emptyset) - \mu(\emptyset) = 0,$$

and if $(E_n)_{n\in\mathbb{N}}$ are pairwise disjoint and $E = \bigsqcup_{n\in\mathbb{N}} E_n$, then

$$\nu(E) = \sum_{n=1}^{\infty} \mu(E_n \cap A) - \sum_{m=1}^{\infty} \mu(E_m \cap B) \overset{(*)}{=} \sum_{n=1}^{\infty} (\mu(E_n \cap A) - \mu(E_n \cap B)) = \sum_{n=1}^{\infty} \nu(E_n).$$

In step $(*)$, the reordering of the sum is justified by noting that μ is a finite measure, so all of the sums involved are absolutely convergent and can be freely rearranged by the Riemann rearrangement theorem.

(b) We claim that $(A, X \setminus A)$ is a Hahn decomposition of ν. Indeed, if $E \in \mathcal{B}$ with $E \subseteq A$, then

$$\nu(E) = \mu(E \cap A) - \mu(E \cap B) = \underbrace{\mu(E \cap A) - \mu(E \cap (A \cap B))}_{\geq 0} - \mu(\underbrace{E \cap (B \setminus A)}_{\emptyset}) \geq 0.$$

Moreover, if $E \in \mathcal{B}$ and $E \subseteq X \setminus A$, then

$$\nu(E) = \mu(\underbrace{E \cap A}_{\emptyset}) - \mu(E \cap B) = -\mu(E \cap B) \leq 0.$$

(c) Suppose $\mu(N) = 0$. We want to show that N is a ν-null set. Let $E \in \mathcal{B}$ with $E \subseteq N$. Then $\nu(E) = \mu(\underbrace{E \cap A}_{\subseteq N}) - \mu(\underbrace{E \cap B}_{\subseteq N}) = 0$, so N is ν-null as desired.

(d) We claim $\frac{d\nu}{d\mu} = \mathbb{1}_A - \mathbb{1}_B$ μ-a.e. To see this, let $E \in \mathcal{B}$. Then

$$\int_E (\mathbb{1}_A - \mathbb{1}_B)\, d\mu = \int_X (\mathbb{1}_A - \mathbb{1}_B)\mathbb{1}_E\, d\mu = \int_X (\mathbb{1}_{A\cap E} - \mathbb{1}_{B\cap E})\, d\mu = \mu(A \cap E) - \mu(B \cap E) = \nu(E),$$

so $d\nu = (\mathbb{1}_A - \mathbb{1}_B)\, d\mu$ as claimed.

Problem 13. Let X be an LCH space, and let μ be a Radon measure on X. Show that there exists a closed set $C \subseteq X$ with the following two properties:

(i) $\mu(X \setminus C) = 0$, and
(ii) if $U \subseteq X$ is open and $U \cap C \neq \emptyset$, then $\mu(U) > 0$.

[Note: The set C is called the (topological) *support* of the measure μ.]

Solution: Let $\mathcal{U} = \{U \subseteq X \text{ open} : \mu(U) = 0\}$, and define $C = X \setminus \bigcup_{U\in\mathcal{U}} U$. Then C is a closed subset, and we will check that it has the desired properties.

(i) Let $V = X \setminus C = \bigcup_{U\in\mathcal{U}} U$. We want to show $\mu(V) = 0$. Since V is open and μ is a Radon measure, we have

$$\mu(V) = \sup_{K\subseteq V \text{ compact}} \mu(K)$$

by inner regularity of Radon measures on open sets. It therefore suffices to show: if $K \subseteq V$ is compact, then $\mu(K) = 0$. Let $K \subseteq V$ be compact. By compactness, we may find finitely many $U_1, \ldots, U_n \in \mathcal{U}$ such that $K \subseteq \bigcup_{j=1}^n U_j$. Then by monotonicity and (countable) subadditivity of μ,

$$\mu(K) \leq \sum_{j=1}^{n} \mu(U_j) = 0$$

since $U_j \in \mathcal{U}$ for each $j \in \{1, \ldots, n\}$. This proves (i).

(ii) Suppose $U \subseteq X$ is open and $U \cap C \neq \emptyset$. Then $U \notin \mathcal{U}$, so $\mu(U) > 0$.

Problem 14. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Suppose ν_1, ν_2 are positive measures on $(X, \mathcal{B}, \mu)$ with $\nu_1(X) = \nu_2(X) = 1$ and $\nu_1, \nu_2 \ll \mu$. Show

$$\sup_{E \in \mathcal{B}} (\nu_1(E) - \nu_2(E)) = \frac{1}{2} \int_X \left| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right| \, d\mu.$$

Solution: Let $\nu = \nu_1 - \nu_2$. Finiteness of the measures ν_1 and ν_2 ensures that ν is a well-defined signed measure. Then the left hand side is equal to $\nu^+(E)$ and the right hand side is equal to

$$\frac{1}{2} \int_X \left| \frac{d\nu}{d\mu} \right| \, d\mu. \tag{6}$$

We showed in the last homework assignment that $\left| \frac{d\nu}{d\mu} \right|$ is the Radon–Nikodym derivative of $|\nu|$ with respect to μ. Hence, (6) is equal to

$$\frac{1}{2} \int_X \frac{d|\nu|}{d\mu} \, d\mu = \frac{1}{2} |\nu|(X).$$

Thus, our goal is to show $\nu^+(X) = \frac{1}{2}|\nu|(X)$. Since $|\nu| = \nu^+ + \nu^-$, this is equivalent to showing $\nu^+(X) = \nu^-(X)$, but this follows immediately from the assumption $\nu_1(X) = \nu_2(X) < \infty$:

$$\nu^+(X) - \nu^-(X) = (\nu^+ - \nu^-)(X) = \nu(X) = (\nu_1 - \nu_2)(X) = \nu_1(X) - \nu_2(X) = 0.$$

Problem 15. Let $A, B \subseteq [0,1)$ be Lebesgue-measurable sets. For each $t \in [0,1)$, let $B_t = \{b + t \bmod 1 : b \in B\}$. Show that there exists $t \in [0,1)$ such that $\lambda(A \cap B_t) \geq \lambda(A)\lambda(B)$.

Solution: We will show that the average size of the intersection $\lambda(A \cap B_t) = \lambda(A)\lambda(B)$. Indeed, by Tonelli's theorem and invariance of Lebesgue measure under translation and reflection, we have

$$\begin{aligned}
\int_0^1 \lambda(A \cap B_t) \, dt &= \int_0^1 \int_0^1 \mathbb{1}_{A \cap B_t}(x) \, dx \, dt \\
&= \int_0^1 \int_0^1 \mathbb{1}_A(x) \mathbb{1}_{B_t}(x) \, dx \, dt \\
&= \int_0^1 \int_0^1 \mathbb{1}_A(x) \mathbb{1}_B(x - t \bmod 1) \, dx \, dt \\
&= \int_0^1 \mathbb{1}_A(x) \left(\int_0^1 \mathbb{1}_B(x - t \bmod 1) \, dt \right) \, dx \\
&= \int_0^1 \mathbb{1}_A(x) \, dx \int_0^1 \mathbb{1}_B(t) \, dt \\
&= \lambda(A)\lambda(B).
\end{aligned}$$

Subtracting $\lambda(A)\lambda(B)$ from both sides, we obtain

$$\int_0^1 (\lambda(A \cap B_t) - \lambda(A)\lambda(B)) \, dt = 0,$$

so there exists $t \in [0,1)$ such that $\lambda(A \cap B_t) - \lambda(A)\lambda(B) \geq 0$. That is, $\lambda(A \cap B_t) \geq \lambda(A)\lambda(B)$.

Problem 16. Let X be a compact metric space, and let $T : X \to X$ be a continuous function. We say that a probability measure $\mu : \mathrm{Borel}(X) \to [0, \infty]$ is *T-invariant* if $\mu(T^{-1}E) = \mu(E)$ for every $E \in \mathrm{Borel}(X)$. Denote by $\mathcal{M}(X, T)$ the space of T-invariant Borel probability measures on X.

(a) Show that $\mathcal{M}(X, T)$ is a convex set.

Given a convex set C, a point $x \in C$ is an *extreme point* if the only solution to $x = ty + (1-t)z$ for $t \in (0,1)$ and $y, z \in C$ is $y = z = x$.

(b) Let $\mu \in \mathcal{M}(X,T)$. Show that the following are equivalent:
   (i) μ is an extreme point of $\mathcal{M}(X,T)$.
   (ii) if $E \in \mathcal{B}$ and $\mu\left(E \triangle T^{-1}E\right) = 0$, then $\mu(E) \in \{0,1\}$.

[**Hint for (ii) $\Longrightarrow$ (i):** Suppose μ satisfies (ii), and write $\mu = t\nu_1 + (1-t)\nu_2$ with $\nu_1, \nu_2 \in \mathcal{M}(X,T)$ and $t \in (0,1)$. Let $f = \frac{d\nu_1}{d\mu}$ and consider $E = \{f < 1\}$. Show that $\int_{E \setminus T^{-1}E} f \, d\mu = \int_{T^{-1}E \setminus E} f \, d\mu$ and deduce that $f = 1$ a.e.]

A measure satisfying (ii) is called *ergodic*. Let $\mathcal{E}(X,T)$ denote the set of ergodic T-invariant Borel probability measures on X.

(c) Suppose $\mu, \nu \in \mathcal{E}(X,T)$ and $\mu \neq \nu$. Show that $\mu \perp \nu$.

Solution: **(a)** Let $\mu, \nu \in \mathcal{M}(X,T)$ and $t \in [0,1]$. Let $\rho = t\mu + (1-t)\nu$. We want to show $\rho \in \mathcal{M}(X,T)$. Let $E \in \text{Borel}(X)$. Then

$$\rho\left(T^{-1}E\right) = t\mu\left(T^{-1}E\right) + (1-t)\nu\left(T^{-1}E\right) = t\mu(E) + (1-t)\nu(E) = \rho(E)$$

by T-invariance of μ and ν.

(b) Suppose μ is an extreme point of $\mathcal{M}(X,T)$. We want to show μ is an ergodic measure. Let $E \in \mathcal{B}$ such that $\mu(E \triangle T^{-1}E) = 0$ and suppose for contradiction that $t = \mu(E) \in (0,1)$. Define probability measures $\mu_1, \mu_2 : \text{Borel}(X) \to [0,1]$ by

$$\mu_1(A) = t^{-1}\mu(A \cap E) \qquad \text{and} \qquad \mu_2(A) = (1-t)^{-1}\mu(A \setminus E).$$

Note that

$$\mu_1\left(T^{-1}A\right) = t^{-1}\mu\left(T^{-1}A \cap E\right) \overset{(*)}{=} t^{-1}\mu\left(T^{-1}A \cap T^{-1}E\right) \overset{(**)}{=} t^{-1}\mu(A \cap E) = \mu_1(A),$$

where in $(*)$ we use $\mu(E \triangle T^{-1}E) = 0$ and in $(**)$ we use T-invariance of μ. Similarly,

$$\mu_2\left(T^{-1}A\right) = (1-t)^{-1}\mu\left(T^{-1}A \setminus E\right) = (1-t)^{-1}\mu\left(T^{-1}A \setminus T^{-1}E\right) = (1-t)^{-1}\mu(A \setminus E) = \mu_2(A).$$

Thus, $\mu_1, \mu_2 \in \mathcal{M}(X,T)$, so $\mu = t\mu_1 + (1-t)\mu_2$ expresses μ as a nontrivial linear combination of elements of $\mathcal{M}(X,T)$. This contradicts the assumption that μ is an extreme point. Thus, $\mu(E) \in \{0,1\}$ and μ is an ergodic measure.

Conversely, suppose μ is ergodic and write $\mu = t\nu_1 + (1-t)\nu_2$ with $\nu_1, \nu_2 \in \mathcal{M}(X,T)$ and $t \in (0,1)$. We want to show $\nu_1 = \nu_2 = \mu$. Note that $\nu_1, \nu_2 \ll \mu$. Let $f = \frac{d\nu_1}{d\mu}$, and consider $E = \{f < 1\}$. Then (as in the proof of the Radon–Nikodym theorem) the pair $(E, X \setminus E)$ is a Hahn decomposition of the signed measure $\mu - \nu_1$. Since μ and ν_1 are both T-invariant, the pair $(T^{-1}E, X \setminus T^{-1}E)$ is also a Hahn decomposition of $\mu - \nu_1$. Hence, $E \triangle T^{-1}E$ is a $(\mu - \nu_1)$-null set by (the uniqueness part of) the Hahn decomposition theorem. In particular, $E \triangle T^{-1}E$ is μ-null. By the ergodicity assumption, it follows that $\mu(E) \in \{0,1\}$. If $\mu(E) = 1$, then $f < 1$ μ-a.e., so $\nu_1(X) = \int_X f \, d\mu < \mu(X) = 1$, which is impossible, since ν_1 is a probability measure. Hence, $\mu(E) = 0$, so $f \geq 1$ μ-a.e. A similar argument shows $f \leq 1$ μ-a.e, so $f = 1$ μ-a.e. That is, $\nu_1 = \mu$. This proves that μ is an extreme point of $\mathcal{M}(X,T)$.

(c) Let $\mu, \nu \in \mathcal{E}(X,T)$ with $\mu \neq \nu$. Let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of ν with respect to μ. Write $X = A \sqcup B$ with $A, B \in \mathcal{B}$ such that $\nu_a(B) = 0$ and $\nu_s(A) = 0$. Then (A, B) is a Hahn decomposition for the signed measure $\infty \cdot \mu - \nu$ and $\nu_a(E) = \nu(E \cap A)$ and $\nu_s(E) = \nu(E \cap B)$ for $E \in \mathcal{B}$. By T-invariance of μ and ν, the pair $(T^{-1}A, T^{-1}B)$ is also a Hahn decomposition of $\infty \cdot \mu - \nu$, so $A \triangle T^{-1}A = B \triangle T^{-1}B$ is a null set for $\infty \cdot \mu - \nu$. In particular, ν_a and ν_s are T-invariant measures. Suppose for contradiction that $\nu_a \neq 0$. Then $\rho = \frac{\nu_a}{\nu_a(X)} \in \mathcal{M}(X,T)$ and $\rho \ll \nu$ and $\rho \ll \mu$. By the argument in the last paragraph in the proof of part (b), it follows that $\rho = \nu$ and $\rho = \mu$. Hence, $\nu = \mu$. This is a contradiction, so we must have $\nu_a = 0$. That is, $\nu = \nu_s$ is singular to μ.